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# Group Cooperation in Intergroup Conflicting Networks: An Evolutionary Game Approach ⋆

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**Abstract:** Most intergroup conflicts arise from rivalry over limited resources, malicious disturbance, or hostile attitudes; therefore, it is critical to investigate individuals' behaviors involved in intergroup conflicting contexts. Focusing on cooperation issues among individuals, in this study we establish an evolutionary game framework for analyzing cooperation and conflicts that arise within and inter-groups of intergroup conflicting networks. We first model the intergroup conflicting networked evolutionary games (ICNEGs). Then, we analyze the ICNEGs and prove that the evolution of ICNEGs can be expressed as a logical dynamic system. Finally, we apply the obtained results to a simplified Israeli-Palestinian conflict scenario. Our case study demonstrates that only by adopting suitable initial strategy profiles can a certain scale of group cooperation be continuously generated without suffering casualties.



*Keywords:* Coupled networks, intergroup conflicting, cooperation, evolutionary game approach.

## 1. INTRODUCTION

As a powerful tool for analyzing agents' behavior in complex networks, networked evolutionary games have attracted increasing attention (Cheng et al. (2015); Chen et al. (2019); Zhao et al. (2018); Zhu et al. (2022)). For instance, an evolutionary game model was proposed to analyze the structural conflict of signed networks by Tan et al. (2016). Riehl et al. (2017) discussed the optimal control problem of networked evolutionary games. Cooperation is ubiquitous in interconnected and complex networks, such as social, economic, and biological systems (Nowak et al. (1992); Ramazi et al. (2015)). In reality, most network individuals are selfish and try to maximize their benefits while pursuing cooperation. Thus, how cooperation emerges from networked evolutionary games is a focal issue for study.

In the past two decades, most studies on networked evolutionary games focus on single network structures (Zhu et al. (2023)). However, in order to better simulate real-world networked systems, interaction and intergroup conflicts among networks need to be considered (Guo et al. (2021); Danziger et al. (2022)). For instance, a kind of networked evolutionary game with coupled social groups was investigated by Guo et al. (2021). Bornstein (2003) established a coherent framework to investigate the cooperation and conflicts that arise within and between groups. Notably, networks with intergroup conflicts are susceptible to external attacks. Concerning the attack issue, Hao et al. (2020) offered fourteen edge-attack techniques utilizing the network properties. Chen et al. (2019) introduced six primary categories of malicious attacks on networks. Considering rivalry for limited resources or malicious disturbance between networks, it is important to explore how group cooperation arises from networked evolutionary games under intergroup conflicts with attacking effects.

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This paper studies group cooperation in intergroup conflicting networks, and is organized as follows. Section 2 proposes the conflicting game model. Section 3 introduces intergroup conflicting networked evolutionary games (ICNEGs). Section 4 establishes the mathematical model of ICNEGs. An illustrative case is simulated and discussed in Section 5. Conclusions are drawn in Section 6.

Notation: $\mathcal{D}_k := \{1, \cdots, k\}$. $Col_i(A)$ is the $i$-th column of matrix $A$. $\Delta_n := \{\delta_n^k = Col_k(I_n) : k = 1, \cdots, n\}$. $L = [\delta_m^{i_1}, \cdots, \delta_m^{i_n}]$ is denoted by $L = \delta_m[i_1, \cdots, i_n]$. $\mathbf{1}_\ell = [1, \cdots, 1]^T \in \mathbb{R}^\ell$. $E_\ell = \mathbf{1}^T_{2^{\ell-1}} \otimes I_2 \otimes \mathbf{1}^T_{2^{n-\ell}} \in \mathcal{M}_{2\times 2^n}$, $\forall\, \ell \in \{1, \cdots, n\}$. "$\ltimes$" is the semi-tensor product of matrices (Cheng et al. (2011)). "$*$" is the Khatri-Rao product of matrices.

## 2. CONFLICTING GAME MODEL

In this section, a two-layer coupled network and the corresponding attack mechanism are introduced. Our aim

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is to construct a mathematical framework to analyze the group cooperation of intergroup conflicting networks.

In this framework, every network player adaptively updates its strategy and obtains a payoff by communicating with its neighbors in the same network. Suppose that among these players, there exists an attacker type of player who does not communicate with its neighbors but monitors the players on the opponent's network in real-time.

The proposed two-layer coupled network $\Xi := (\Xi^1, \Xi^2, E)$ is consists of the following factors: i) two simple undirected networks $\Xi^i := (N^i, E^i)$, $i = 1, 2$, where $N^i = N_0^i \cup \{q^i\} = \{p_1^i, \cdots, p_\star^i, \cdots, p_{n^i}^i\} \cup \{q^i\}$, $n^i > 1$, is the player set, $E^i \subseteq N_0^i \times N_0^i$ is the edge set, $q^i$ is the attacker in $\Xi^i$, and $p_\star^i$ be the targeted player in $\Xi^i$; ii) The set of directed edges $E = \{(q^1, p_\star^2), (q^2, p_\star^1)\} \subseteq (N^1 \times N^2) \cup (N^2 \times N^1)$.

*Definition 1.* For network $\Xi^i$, $i \in \{1, 2\}$, if there exists a path from player $p_j^i \in N^i$ to player $\tilde{p}_j^i \in N^i$ with length not longer than $\ell \in \mathbb{N}$, then $\tilde{p}_j^i$ is said to be an $\ell$-neighbor of player $p_j^i$. Denote the set of $\ell$-neighbors of player $p_j^i$ by $U_\ell(p_j^i)$. Let $U_1(p_j^i) := U(p_j^i)$ and $U_0(p^i) := \{p^i\}$.

Next, the intergroup conflict mechanism in the two-layer coupled network $\Xi$ is characterized.

1) **Attack mechanism** $\Omega_{\mathcal{Q}}$: Consider network $\Xi^i$, $i \in \{1, 2\}$. Player $p_j^i \in N_0^i$ adaptively updates its strategy and obtains a payoff at each time $t \in \mathbb{N}$, denoted as $c_{p_j^i}(t)$, by communicating with its neighbors in the same network. Attacker player $q^i \in N^i$ will carry out the attack strategy $s_a$ onto the targeted player $p_\star^{\bar{i}} \in N^{\bar{i}}$, $\bar{i} = \{1, 2\} \setminus \{i\}$, at time $t + 1$ if $c_{p_\star^{\bar{i}}}(t) > \mathcal{Q}_i \in \mathbb{R}$; otherwise, $q^i$ will hold the monitoring strategy $s_m$.

Denote the health point of player $p^i \in N^i$, $i \in \{1, 2\}$, at time $t \in \mathbb{N}$ by $\Psi_{p^i}(t)$. Initially, $\Psi_{p^i}(0) = 1$.

2) **Damage cost and attack cost:** For targeted player $p_\star^i \in N_0^i$ who attacked by $q^{\bar{i}} \in N^{\bar{i}}$, define $c_D^i \in (0, 1]$ as the damage cost of $p_\star^i$, and $c_A^{\bar{i}} \in (0, c_D^i)$ the attack cost of $q^{\bar{i}}$.

*Definition 2.* For player $p^i \in N^i$, $i \in \{1, 2\}$, if there exists $T_d \in \mathbb{N}$ such that $\Psi_{p^i}(T_d) = 0$, player $p^i$ will be dead and withdrawn from network $\Xi^i$ after time $T_d$.

## 3. INTERGROUP CONFLICTING NETWORKED EVOLUTIONARY GAMES

In this section, the concept of ICNEGs is introduced.

*Definition 3.* (Cheng et al. (2015)) The finite noncooperative game $G_0 := (N_0, (S_p)_{p \in N_0}, (c_p)_{p \in N_0})$, with player set $N_0 = \{p_1, p_2\}$, strategy set $S_p$, $p \in N_0$, and payoff function $c_p$, $p \in N_0$, is said to be a symmetric matrix game, if $S_{p_1} = S_{p_2} = S_0 = \{s_\alpha, s_\beta\}$, and $c_{p_1}(s_\alpha, s_\beta) = c_{p_2}(s_\beta, s_\alpha)$, $\forall\, s_\alpha, s_\beta \in S_0$.

In a symmetric matrix game, both players have two strategies, commonly referred to as "Cooperation (C)" and "Defection (D)". Here, we consider the positive-symmetric coordination matrix game, which has wide applications in networked evolutionary dynamics (Riehl et al. (2018)).

*Definition 4.* A symmetric matrix game $G_0 := (N_0, S_0 = \{C, D\}, (c_p)_{p \in N_0})$ is said to be a positive-symmetric coordination matrix game, if the payoff function of player $p \in N_0$ is determined by the following matrix:

$$\begin{array}{c c} & \begin{array}{cc} \mathrm{C} & \mathrm{D} \end{array} \\ \begin{array}{c} \mathrm{C} \\ \mathrm{D} \end{array} & \begin{bmatrix} a_{11}^p & a_{12}^p \\ a_{21}^p & a_{22}^p \end{bmatrix} \end{array}, \; a_{11}^p > a_{21}^p > 0, \; a_{22}^p > a_{12}^p > 0. \tag{1}$$

Now, we introduce the concept of ICNEGs.

*Definition 5.* An ICNEG, $\mathbb{G} := (\Xi, C_{[G_0^1, G_0^2]}, S, \Omega_{\mathcal{Q}}, \Pi)$, is consists of

i) two-layer coupled network $\Xi$;
ii) payoff set $C_{[G_0^1, G_0^2]} := \{c_{p_j^i} : p_j^i \in N_0^i,\ i = 1, 2\}$, where $c_{p_j^i}$ is determined by $p_j^i$ playing positive-symmetric coordination matrix game $G_0^i$ with each $\tilde{p}_j^i \in U(p_j^i)$;
iii) strategy set $S := S_0 \cup \{s_a, s_m\}$, where $S_0 = \{C, D\}$;
iv) attack mechanism $\Omega_{\mathcal{Q}}$, where $\mathcal{Q} = \{\mathcal{Q}_1, \mathcal{Q}_2\}$, and $\mathcal{Q}_i$, $i \in \{1, 2\}$, is the preset threshold of $\Xi^i$;
v) strategy updating rule set $\Pi = \{\Pi_1, \Pi_2\}$, where $\Pi_i$, $i \in \{1, 2\}$, is the strategy updating rule of $\Xi^i$.

*Remark 1.* Based on Definition 2 and Definition 5, the two-layer coupled network $\Xi$ of $\mathbb{G}$ is a time-varying network.

The dynamics of ICNEG, $\mathbb{G}$, can be expressed as follows:

$$\begin{cases} \Xi(t+1) &= \xi\Big(\Xi(0), s(0), \cdots, s(t)\Big); \\ s_{p_j^i}(t+1) &= h_{p_j^i}\Big((s_{p_j^i}(t), s_{\tilde{p}_j^i}(t), c_{\tilde{p}_j^i}(t)) : \\ & \qquad \tilde{p}_j^i \in U^t(p_j^i)\Big),\ p_j^i \in \Xi(t+1); \\ s_{q^i}(t+1) &= l_{q^i}\Big(c_{p_\star^{\bar{i}}}(t) : (q^i, p_\star^{\bar{i}}) \in E(t)\Big); \\ s_{q^i}(0) &= s_m, \quad i = 1, 2,\ j = 1, \cdots, n^i,\ t \in \mathbb{N}, \end{cases} \tag{2}$$

where $s_i(t) := (s_{p_1^i}(t), \cdots, s_{p_{n^i}^i}(t))$, $i \in \{1, 2\}$, is the strategy profile of $\Xi^i$ at time $t$, $s(t) := (s_1(t), s_2(t), s_{q^1}(t), s_{q^2}(t))$, and $U^t(p_j^i)$ is the set of 1-neighbors of player $p_j^i$ at time $t$.

Denote the strategy of player $p^i \in N^i$, $i \in \{1, 2\}$, at time $t \in \mathbb{N}$ with $s_i(0) := s_{i_0}$ by $s_{p^i}(t; s_i(0) := s_{i_0})$. Similarly, denote the strategy profile of network $\Xi^i$, $i \in \{1, 2\}$, at time $t \in \mathbb{N}$ with $s_i(0) := s_{i_0}$ by $s_i(t; s_i(0) := s_{i_0})$.

*Definition 6.* Consider an ICNEG, $\mathbb{G}$. For any $i \in \{1, 2\}$, $\Xi^i$ is said to achieve finite-time strong cooperation, if there exists $T_i \in \mathbb{N}$ such that $s_{p_j^i}(t; s_i(0) = s_{i_0}) = C$, $\forall\, p_j^i \in N_0^i$, holds for any integers $t \geqslant T_i$ with any $s_i(0) := s_{i_0}$.

*Definition 7.* Consider an ICNEG, $\mathbb{G}$. For any $i \in \{1, 2\}$, $\Xi^i$ is said to achieve finite-time weak cooperation with $s_i(0) := s_{i_0}$, if there exists $T_i \in \mathbb{N}$ such that $s_{p_j^i}(t; s_i(0) = s_{i_0}) = C$, $\forall\, p_j^i \in N_0^i \setminus \{p_\star^i\}$ and $\Psi_{p_\star^i}(t) = 0$ hold for all integers $t \geqslant T_i$.

## 4. MATHEMATICAL MODEL OF ICNEGS

In this section, we establish an algebraic mathematical model for ICNEGs.

### *4.1 Structure and Strategy Dynamics*

To begin with, we present the following hypothesis.

*Hypothesis 2.* Consider an ICNEG, $\mathbb{G}$. For any $i \in \{1, 2\}$, player $p_\star^i$ who withdraws from the network $\Xi^i(t_D - 1)$ due to $\Psi_{p^i_\star}(t_D) = 0$ will be considered as a virtual player of network $\Xi^i$ after time $t_D$. Henceforth, the virtual player $p^i_\star$ takes the dead strategy, denoted as $s_D$, that is, $s_{p^i_\star}(t) = s_D$, $\forall\, t \geqslant t_D$, $t \in \mathbb{N}$.

Based on Hypothesis 2, the structure of $\Xi(t)$, $t \in \mathbb{N}$, can be handled on the fixed network, that is, the structure dynamics of ICNEG (2) can be converted to

$$\Xi(t) = \Xi(0) = \Xi,\ \forall\, t \in \mathbb{N}. \tag{3}$$

Furthermore, according to Hypothesis 2, for any $i \in \{1,2\}$, the strategy set of player $p^i_\star \in N^i_0$ is expended to $\tilde{S}_0 := S_0 \cup \{s_D\}$. Moreover, denote $s^i_e = (C)^{n^i}$ and $s^i_\omega = \ltimes_{j=1}^{n^i} s_{p^i_j}$, where $s_{p^i_j} = C$, $p^i_j \in N^i_0 \setminus \{p^i_\star\}$, and $s_{p^i_\star} = s_D$.

Next, we describe the strategy dynamics of ICNEG (2).

According to ICNEG (2) and attack mechanism $\Omega_{\mathcal{Q}}$, for any $i \in \{1,2\}$, the strategy dynamics of player $p^i_j \in N^i_0$ can be expressed as

$$s_{p^i_j}(t+1) = \begin{cases} s_D, & \text{if } p^i_j = p^i_\star \text{ and } \Psi_{p^i_\star}(t+1) = 0; \\ f_{p^i_j}\Big((s_{\tilde{p}^i_j}(t), c_{\tilde{p}^i_j}(t)) : \tilde{p}^i_j \in U(p^i_j)\Big), & \text{otherwise}, \end{cases} \tag{4}$$

where $f_{p^i_j}$ is a mapping that is determined by the strategy updating rule $\Pi_i$.

For any $i \in \{1,2\}$, since $c^i_D > c^i_A$, the strategy dynamics of attacker player $q^i$ can be expressed as

$$s_{q^i}(t+1) = \begin{cases} s_a, & \text{if } c_{p^{\bar{i}}_\star}(t) > \mathcal{Q}; \\ s_m, & \text{otherwise}. \end{cases} \tag{5}$$

### 4.2 Algebraic Expression

Now, we convert the strategy dynamics of ICNEGs into an equivalent algebraic form.

*Definition 8.* The total payoff of player $p^i_j \in N^i_0$, $i \in \{1,2\}$, at time $t \in \mathbb{N}$, is

$$c_{p^i_j}(t) := \sum_{\tilde{p}^i_j \in U(p^i_j) \setminus \{p^i_j\}} c^{\tilde{p}^i_j}_{p^i_j}\Big(s_{p^i_j}(t), s_{\tilde{p}^i_j}(t)\Big), \tag{6}$$

where $c^{\tilde{p}^i_j}_{p^i_j}\Big(s_{p^i_j}(t), s_{\tilde{p}^i_j}(t)\Big)$, $s_{p^i_j}(t) \neq s_D$, $s_{\tilde{p}^i_j}(t) \neq s_D$, is the payoff of player $p^i_j$ playing game $G^i_0$ with player $\tilde{p}^i_j \in U(p^i_j)$ at time $t \in \mathbb{N}$, and $c^{\tilde{p}^i_j}_{p^i}(s_{p^i_j}(t), s_{\tilde{p}^i_j}(t)) = 0$ when $s_{p^i_j}(t) = s_D$ or $s_{\tilde{p}^i_j}(t) = s_D$.

First, based on Definition 8, (4) and (5), we obtain the following results.

*Proposition 3.* Consider an ICNEG, $\mathbb{G}$. For any $i \in \{1,2\}$, the strategy dynamics of player $p^i_j \in N^i_0$, $p^i_j \neq p^i_\star$, can be obtained as

$$s_{p^i_j}(t+1) = g_{p^i_j}\Big((s_{p^i_j}(t), s_{\tilde{p}^i_j}(t)) : \tilde{p}^i_j \in U_2(p^i_j)\Big). \tag{7}$$

*Proposition 4.* Consider an ICNEG, $\mathbb{G}$. For any $i \in \{1,2\}$, the strategy dynamics of player $p^i_\star$ can be obtained as

$$\begin{aligned} s_{p^i_\star}(t+1) = m^t_{p^i_\star}\Big((&s_{p^i_\star}(t), s_{\tilde{p}^i}(t), s_{q^{\bar{i}}}(0), \cdots, \\ &s_{q^{\bar{i}}}(t+1)) : \tilde{p}^i \in U_2(p^i_\star)\Big),\ t \in \mathbb{N}. \end{aligned} \tag{8}$$

Denote the strategies $C$, $D$, and $s_D$ by 1, 2 and 3. Now, we transform (7) and (8) into the equivalent algebraic forms. To this end, for any $i \in \{1,2\}$ and any $p^i_j \in N^i_0$, denote strategy $s_{p^i_j} \in \tilde{S}_0 = \mathcal{D}_3$ by vector $\delta_3^{s_{p^i}} \in \Delta_3$, and for attacker player $q^i$, denote strategy $s_a$ by $\delta^1_2 \in \Delta_2$, and monitoring strategy $s_m$ by $\delta^2_2 \in \Delta_2$, respectively.

Based on Proposition 3 above and Theorem A.7 in Cheng et al. (2015), for any $i \in \{1,2\}$, the strategy of player $p^i_j \in N^i_0$, $p^i_j \neq p^i_\star$, at time $t+1$, $t \in \mathbb{N}$, can be obtained as

$$s_{p^i_j}(t+1) = G_{p^i_j} \ltimes_{\tilde{p}^i_j \in U_2(p^i_j)} s_{\tilde{p}^i_j}(t), \tag{9}$$

where $G_{p^i_j} \in \mathcal{M}_{3 \times 3^{\mathbb{U}^i_j}}$, $\mathbb{U}^i_j := |U_2(p^i_j)|$, is the structure matrix of $g_{p^i_j}$ in (7).

Furthermore, let $U_2(p^i_j) := \Big\{p^i_{j_1}, \cdots, p^i_{j_{\mathbb{U}^i_j}}\Big\}$ and $j_1 < \cdots < j_{\mathbb{U}^i_j}$. Then, one obtains

$$\begin{aligned} s_{p^i_j}(t+1) &= G_{p^i_j} \Lambda^{p^i_j}_1 [I_{3^{j_1}} \otimes \Lambda^{p^i_j}_2 [I_{3^{j_2 - j_1}} \\ &\otimes \Lambda^{p^i_j}_3 [\cdots] \cdots]] s_i(t) := \tilde{G}_{p^i_j} s_i(t), \end{aligned} \tag{10}$$

where $\Lambda^{p^i_j}_\alpha := (\mathbf{1}^T_{3^{j_\alpha - j_{\alpha-1} - 1}} \otimes I_3)$, $\forall\ \alpha \in \mathcal{D}_{\mathbb{U}^i_j - 1}$ and $\Lambda^{p^i_j}_{\mathbb{U}^i_j} := \Big(\mathbf{1}^T_{3^{j_{\mathbb{U}^i_j} - j_{\mathbb{U}^i_j - 1} - 1}} \otimes I_3 \otimes \mathbf{1}^T_{3^{n^i - j_{\mathbb{U}^i_j}}}\Big)$.

Let $\mathbb{G}^i := \tilde{G}_{p^i_1} * \cdots * \tilde{G}_{p^i_{n^i}}$. Then, for any $i \in \{1,2\}$, when $\Psi_{p^i_\star}(t+1) > 0$, $t \in \mathbb{N}$, the strategy profile of network $\Xi^i$ at time $t+1$ can be expressed as

$$s_i(t+1) := \mathbb{G}^i s_i(t). \tag{11}$$

Next, we analyze the strategy dynamics of $p^i_\star$, $i \in \{1,2\}$.

Similarly, based on Proposition 4, for any $i \in \{1,2\}$, the strategy of player $p^i_\star \in N^i$ at time $t+1$, $t \in \mathbb{N}$, can be expressed as

$$s_{p^i_\star}(t+1) = M^t_{p^i_\star} \ltimes_{\tau=0}^{t+1} s_{q^{\bar{i}}}(\tau) \ltimes_{\tilde{p}^i \in U_2(p^i_\star)} s_{\tilde{p}^i}(t), \tag{12}$$

where $M^t_{p^i_\star} \in \mathcal{M}_{3 \times \theta_{p^i_\star}(t)}$ is the structure matrix of $m^t_{p^i_\star}$ at time $t$ in (8), $\theta_{p^i_\star}(t) := 2^{t+2} \times 3^{\mathbb{U}^i_\star}$, and $\mathbb{U}^i_\star := |U_2(p^i_\star)|$.

### 4.3 Mathematical Framework of ICNEGs

In this subsection, we establish a mathematical framework to analyze the cooperation and intergroup conflicts of ICNEGs.

For any $i \in \{1,2\}$, let $c^i_D = \frac{1}{\lambda_D}$, $\lambda_D \in \mathbb{N}_+$, and $c^i_A = \frac{1}{\lambda_A}$, $\lambda_A \in \mathbb{N}_+$.

*Theorem 5.* Consider an ICNEG, $\mathbb{G}$. For any $i \in \{1,2\}$, if $\Xi^i$ achieves finite-time weak cooperation with $s_i(0) := s_{i_0}$, then there exists $T^i_\omega \in \{\lambda_D, \cdots, (\lambda_D + 1) \cdot 3^{n^i}\}$ such that $\Xi^i$ achieves finite-time weak cooperation with $s_{i_0}$.

**Proof.** See APPENDIX A.

*Theorem 6.* Consider an ICNEG, $\mathbb{G}$. For any $i \in \{1,2\}$, if $\Xi^i$ achieves finite-time strong cooperation in time $T_i \in \mathbb{N}$, then there exists $T^i_e \in \{1, \cdots, 3^{n^i}\}$ such that $\Xi^i$ achieves finite-time strong cooperation in time $T^i_e$.

**Proof.** See APPENDIX B.

From Theorem 5 and Theorem 6, for any $i \in \{1,2\}$, let

$$s_{q^{\bar{i}}}^{[t]} := \ltimes_{\tau=1}^{t} s_{q^{\bar{i}}}(\tau) \ltimes (s_m)^{|\mathbb{Q}^i|-t} \in \mathbb{R}^{2^{|\mathbb{Q}^i|}}, \qquad (13)$$

where $t \in \mathbb{Q}^i = \{1, \cdots, (\lambda_D+1)\cdot 3^{n^i}\}$, and let

$$\begin{aligned} C^i := \Big\{ \delta_{2^{|\mathbb{Q}^i|}}^{\ell} \in \Delta_{2^{|\mathbb{Q}^i|}} : \exists\, t \in \mathbb{Q}^i \text{ s. t. } |\{l \in \mathcal{D}_t : \\ E_l \delta_{2^{|\mathbb{Q}^i|}}^{\ell} = s_a\}| = \lambda_D;\ E_l \delta_{2^{|\mathbb{Q}^i|}}^{\ell} = s_m,\ l \in \mathbb{Q}^i \setminus \mathcal{D}_t \Big\}. \end{aligned} \qquad (14)$$

Consequently, $s_{p_\star^i}(t+1)$, $t \in \mathbb{Q}_0^i := \{0, \cdots, (\lambda_D+1)\cdot 3^{n^i} - 1\}$, can be obtained as

$$s_{p_\star^i}(t+1) = \widetilde{M}_{p_\star^i} X_{q^{\bar{i}}} s_{q^{\bar{i}}}^{[t+1]} \ltimes_{\tilde{p}^i \in U_2(p_\star^i)} s_{\tilde{p}^i}(t), \qquad (15)$$

where

$$\begin{aligned} Col_\ell(X_{q^{\bar{i}}}) &= \begin{cases} \delta_2^2, & \text{if } s_{q^{\bar{i}}}^{[t+1]} = \delta_{2^{|\mathbb{Q}^i|}}^{\ell} \in C^i; \\ \delta_2^1, & \text{otherwise}, \end{cases} \\ \widetilde{M}_{p_\star^i} &= [G_{p_\star^i}, D_{p_\star^i}] \in \mathcal{M}_{3\times 2\cdot 3^{\mathbb{U}_\star^i}}, \\ D_{p_\star^i} &= \delta_3[3, \cdots, 3]. \end{aligned}$$

Furthermore, let $U_2(p_\star^i) := \left\{p_{j_1}^i, \cdots, p_{j_{\mathbb{U}_\star^i}}^i\right\}$ and $j_1 < \cdots < j_{\mathbb{U}_\star^i}$. Then, one obtains

$$\begin{aligned} s_{p_\star^i}(t+1) &= \widetilde{M}_{p_\star^i} X_{q^{\bar{i}}} \Big( I_{2^{|\mathbb{Q}^i|}} \otimes (\Lambda_1^{p_\star^i} [I_{3^{j_1}} \otimes \Lambda_2^{p_\star^i} \\ &\quad [I_{3^{j_2-j_1}} \otimes \Lambda_3^{p_\star^i} [\cdots] \cdots ]]) \Big) s_{q^{\bar{i}}}^{[t+1]} s_i(t) \\ &:= \mathbb{M}_{p_\star^i} s_{q^{\bar{i}}}^{[t+1]} s_i(t). \end{aligned}$$

Next, for any $i \in \{1,2\}$ and any $p_j^i \in N_0^i \setminus \{p_\star^i\}$, based on (10), the strategy of player $p_j^i$ at time $t+1$, $t \in \mathbb{Q}_0^i$, is obtained as

$$\begin{aligned} s_{p_j^i}(t+1) &= \tilde{G}_{p_j^i} s_i(t) = \tilde{G}_{p_j^i} \Big( \mathbf{1}_{2^{|\mathbb{Q}^i|}}^T \otimes I_{3^{n^i}} \Big) s_{q^{\bar{i}}}^{[t+1]} s_i(t) \\ &:= \mathbb{M}_{p_j^i} s_{q^{\bar{i}}}^{[t+1]} s_i(t). \end{aligned}$$

Let $\mathbb{M}^i = \mathbb{M}_{p_1^i} * \cdots * \mathbb{M}_{p_{n^i}^i}$. For any $i \in \{1,2\}$, the strategy profile of network $\Xi^i$ is expressed as

$$s_i(t+1) = \mathbb{M}^i s_{q^{\bar{i}}}^{[t+1]} s_i(t),\ \ t \in \mathbb{Q}_0^i. \qquad (16)$$

## 5. AN ILLUSTRATIVE EXAMPLE

In this section, we investigate the group cooperation of a simplified Israeli-Palestinian conflict model Selvanathan et al. (2020).

Assume $N_0^1 = \{\text{Nablus, Ramallah, Hebron}\}$ and $N_0^2 = \{\text{Tel Aviv-Yafo, Be'Er Sheva}\}$, where $p_\star^1 = p_2^1 = $ Ramallah and $p_\star^2 = p_2^2 = $ Be'Er Sheva (as shown in Fig. 1). Here, define participating in normative collective action as cooperation strategy $C$, absence from such activities as defection strategy $D$, and the nonnormative collective action as attack strategy $s_a$.

Based on Definition 5, we consider an ICNEG, $\mathbb{G} := (\Xi, C_{[G_0^1, G_0^2]}, S, \Omega_{\mathcal{Q}}, \Pi)$, where

i) two-layer coupled network $\Xi$ (as shown in Fig. 2);
ii) payoff set $C_{[G_0^1,G_0^2]} := \{c_{p_j^i} : p_j^i \in N_0^i, i = 1,2\}$, where the payoff bimatrices of $G_0^1$ and $G_0^2$ are shown in Table 1 and Table 2, respectively;


Fig. 1. A simplified Israeli-Palestinian conflict scenario (Giles (2003))

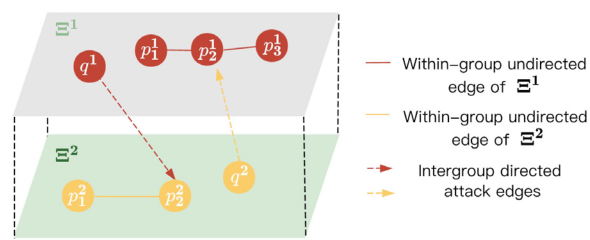


Fig. 2. Two-layer coupled network $\Xi$ of ICNEG

iii) strategy sets $S = S_0 \cup \{s_a, s_m\}$, where $S_0 := \{C, D\}$;
iv) attack mechanism $\Omega_{\mathcal{Q}}$, where $\mathcal{Q}_1 = \mathcal{Q}_2 = 3.7$;
v) $\Pi = \{\Pi_1, \Pi_2\}$, where $\Pi_1$ is myopic best response adjustment with priority, and $\Pi_2$ is unconditional imitation with priority.

Table 1. Payoff bimatrix of $G_0^1$

| $P_1 \setminus P_2$ | C | D |
|---|---|---|
| C | (2,2) | (1,0.2) |
| D | (0.2,1) | (2.5,2.5) |

Table 2. Payoff bimatrix of $G_0^2$

| $P_1 \setminus P_2$ | C | D |
|---|---|---|
| C | (3,3) | (0.3,2) |
| D | (2,0.3) | (6,6) |

Assume $c_D^i = \frac{1}{2}$, $c_A^i = \frac{1}{3}$, $i = 1,2$.

Considering network $\Xi^1$, let $s_{q^2}^{[t]} := \ltimes_{\tau=1}^{t} s_{q^2}(\tau) \ltimes (s_m)^{3^4 - t}$, $t \in \{1, \cdots, 3^4\}$. Based on the proof of Theorem 5, assume

$$s_{q^2}^{[t]} := \ltimes_{\tau=1}^{t} s_{q^2}(\tau) \ltimes (s_m)^{4-t} \in \mathbb{R}^{2^4},\ t \in \mathbb{Q}^1 := \{1, \cdots, 4\}.$$

According to (16), the strategy profile of network $\Xi^1$ at time $t+1$, $t \in \{0,1,2,3\}$, can be expressed as

$$s_1(t+1) = \mathbb{M}^1 s_{q^2}^{[t+1]} s_1(t), \qquad (17)$$

where $\mathbb{M}^1 = \delta_{27}[1,1,3,11,11,12,\cdots,20,23,24,25,26,27]$.

Similarly, for $\Xi^2$, the strategy profile of network $\Xi^2$ at time $t+1$, $t \in \{0,1,2\}$, can be expressed as

$$s_2(t+1) = \mathbb{M}^2 s_{q^1}^{[t+1]} s_2(t),$$

where $\mathbb{M}^2 = \delta_9[1,5,3,5,5,6,\cdots,5,5,6,7,8,9]$.

Take network $\Xi^1$ for example. When the initial strategy profile belongs to $\{(C,C,C),\ (C,C,D),(D,C,C)\}$, network $\Xi^1$ will achieve finite-time weak cooperation with the corresponding initial strategy profile. When the initial strategy profile belongs to

$$\{(C,D,C),\ (C,D,D),\ (D,C,D),(D,D,C)\},$$

player $p^1_\star = p^1_2$ will not be dead in the evolution process. Correspondingly, there exists a strategy profile attractor $(C,D,C) \rightarrow (D,C,D) \rightarrow (C,D,C)$. The simplified model illustrates that only by adopting appropriate initial strategy profiles can a certain scale of group cooperation be generated in $\Xi^1$ without casualties. Specifically, being either overly cowardly (e.g. $(D,D,D)$) or aggressive (e.g. $(C,C,C)$), or targeted city initially cooperating with its neighbors (e.g. $(D,C,C),(C,C,D)$), is not a sensible course of action.

## 6. CONCLUSION

In this paper, from the networked evolutionary game approach, we have established a mathematical framework to study the within-group cooperation of intergroup conflicting networks. The current study has significant ramifications for the development of complex coupled networks associated with conflicts. Further research will be carried out to explore the large-scale intergroup conflicting networks (Li et al. (2021)).

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## Appendix A. PROOF OF THEOREM 5

Since $\Xi^i$, $i \in \{1,2\}$, achieves finite-time weak cooperation in time $T_i \in \mathbb{N}$ with initial strategy profile $s_i(0) := s_{i_0}$, according to Definition 7, it holds that

$$\begin{aligned} s_{p^i_j}(t; s_i(0) = s_{i_0}) = C,\ \forall\ p^i_j \in N^i_0 \setminus \{p^i_\star\}, \\ s_{p^i_\star}(t) = s_D, \forall\ t \geqslant T_i,\ t \in \mathbb{N}. \end{aligned} \tag{A.1}$$

From (A.1), one obtains

$$s_i(t; s_i(0) = s_{i_0}) = s^i_\omega,\ \forall\ t \geqslant T_i,\ t \in \mathbb{N}.$$

On one hand, based on (A.1), it can be proved that there exists $T_D \in \{\lambda_D, \cdots, \lambda_D \cdot 3^{n^i}\}$ such that

$$s_{p^i_\star}(t) = s_D,\ \forall\ t \geqslant T_D,\ t \in \mathbb{N}. \tag{A.2}$$

We prove (A.2) by deducing to contradiction. Assume that

$$s_{p^i_\star}(t) \neq s_D,\ \forall\ t \in \{\lambda_D, \cdots, \lambda_D \cdot 3^{n^i}\}. \tag{A.3}$$

Combining (A.1) with (A.3), there exists $t_D \in \{\lambda_D \cdot 3^{n^i} + 1, \cdots, T_i\}$ such that

$$\begin{aligned} s_{p^i_\star}(t_D - 1) \neq s_D; \\ s_{p^i_\star}(t) = s_D,\ \forall\ t \geqslant t_D,\ t \in \mathbb{N}. \end{aligned} \tag{A.4}$$

Based on (A.4), the health point of $p^i_\star$ at time $t_D$ is $\Psi_{p^i_\star}(t_D) = 0$. Thus, there exist $t_1, \cdots, t_{\lambda_D - 1} \in \{0, \cdots, t_D - 2\}$ such that

$$\begin{aligned} (\mathbb{G}^i)^{t_\ell} s_{i_0} \in \Phi_{p^i_\star},\ \forall\ \ell \in \{1, \cdots, \lambda_D - 1\}, \\ (\mathbb{G}^i)^{t_D - 1} s_{i_0} \in \Phi_{p^i_\star}, \end{aligned} \tag{A.5}$$

where $s_i(0) := s_{i_0}$ and

$$\Phi_{p_\star^i} = \left\{ \ltimes_{j=1}^{n^i} s_{p_j^i} : \sum_{\tilde{p}^i \in U(p_\star^i)\backslash\{p_\star^i\}} c_{p_\star^i}^{\tilde{p}^i}(s_{p_\star^i}, s_{\tilde{p}^i}) > \mathcal{Q} \right\}. \quad \text{(A.6)}$$

Consider the strategy profile trajectory starting from the initial strategy profile $s_{i_0}$. Since (11) is a logical dynamic system, the initial strategy profile $s_{i_0}$ will converge to an attractor, which is the fixed point $s_{s_{i_0}}^E$ of (11) at time $t_{s_{i_0}}^E \in \mathbb{N}$, where $t_{s_{i_0}}^E$ is the minimum integer satisfying

$$s_i(t; s_i(0) = s_{i_0}) = s_{s_{i_0}}^E,$$

or the cycle $\mathcal{S}^{s_{i_0}}$ of (11) at time $t_{s_{i_0}}^{\mathcal{S}} \in \mathbb{N}$, where $t_{s_{i_0}}^{\mathcal{S}}$ is the minimum integer satisfying $s_i(t; s_i(0) = s_{i_0}) \in \mathcal{S}^{s_{i_0}}$.

Consequently, there exist two conditions: i) There exists integer $t_d > t_{s_{i_0}}^E (t_{s_{i_0}}^{\mathcal{S}})$ such that player $p_\star^i$ is dead at time $t_d$; ii) There exists integer $t_d \leqslant t_{s_{i_0}}^E (t_{s_{i_0}}^{\mathcal{S}})$ such that player $p_\star^i$ is dead at time $t_d$.

Consider condition ii). Based on the Cayley-Hamilton theorem, $t_d \leqslant t_{s_{i_0}}^E (t_{s_{i_0}}^{\mathcal{S}}) < 3^{n^i}$ is a contradiction to (A.3).

Consider condition i). Assume that there exist fixed point $s_{s_{i_0}}^E \in \Delta_{3^{n^i}}$ and integer $t_{s_{i_0}}^E$ such that

$$s_i(t_{s_{i_0}}^E; s_i(0) := s_{i_0}) = s_{s_{i_0}}^E. \quad \text{(A.7)}$$

Since there exists integer $t_d > t_{s_{i_0}}^E$ such that $s_{p_\star^i}(t) = s_D$ for any integer $t > t_d$, applying Cayley-Hamilton theorem to $\mathbb{G}^i$ in (11), it is easy to verify that $t_{s_{i_0}}^E < 3^{n^i}$. Furthermore, based on (A.7), one has $t_d \in \{t_{s_{i_0}}^E + 1, \cdots, t_{s_{i_0}}^E + \lambda_D\}$ and

$$s_{p_\star^i}(t) = s_D, \ \forall\, t \geqslant t_d, \ t \in \mathbb{N}. \quad \text{(A.8)}$$

Because $t_{s_{i_0}}^E + \lambda_D < 3^{n^i} + \lambda_D \leqslant \lambda_D \cdot 3^{n^i}$, the contradiction appears.

On the other hand, assume that there exists cycle $\mathcal{S}^{s_{i_0}} := (s_i^1, \cdots, s_i^{\rho_{s_{i_0}}})$ with length $\rho_{s_{i_0}}$ such that the initial strategy profile $s_{i_0}$ converges to $\mathcal{S}^{s_{i_0}}$ in (11) at time $t_{s_{i_0}}^{\mathcal{S}}$. Since $\rho_{s_{i_0}} < 3^{n^i}$, and there exists integer $t_d > t_{s_{i_0}}^{\mathcal{S}}$ such that $s_{p_\star^i}(t) = s_D$ for all integers $t > t_d$, there exists at least one $s_i^\alpha$ of $\mathcal{S}^{s_{i_0}}$ with the corresponding positive integer $t_{s_{i_0}}^{\mathcal{S}} \leqslant t_\alpha < 3^{n^i}$ such that

$$s_i(t_\alpha; s_i(0) = s_{i_0}) = s_i^\alpha \in \Phi_{p_\star^i}. \quad \text{(A.9)}$$

Based on (A.9), there exists $t_d^\alpha \in \{t_\alpha + 1, \cdots, t_\alpha + 1 + (\lambda_D - 1)\rho_{s_{i_0}}\}$ such that

$$s_{p_\star^i}(t) = s_D, \ \forall\, t \geqslant t_d^\alpha, \ t \in \mathbb{N}. \quad \text{(A.10)}$$

Because $t_\alpha + 1 + (\lambda_D - 1)\rho_{s_{i_0}} < \lambda_D \cdot 3^{n^i}$, (A.10) is a contradiction to (A.3).

Based on (A.2), for the initial strategy profile $s_i(0) := s_{i_0}$, denote the strategy profile of $\Xi^i$ at time $T_D$ by $s^{[D]} := \ltimes_{j=1}^{n^i} s_{p_j^i}(T_D)$, where $s_{p_\star^i}(T_D) = s_D$. Because $\Xi^i$ achieves finite-time weak cooperation at time $T_i \in \mathbb{N}$ with $s_i(0) := s_{i_0}$, there exists $\tau_i \in \mathbb{N}$ such that $\Xi^i$ achieves weak strategy consensus at $s^\star$ in finite time $\tau_i < 3^{n^i}$ with $s_i(0) := s^{[D]}$. Therefore, there exists $T_\omega^i \in \{\lambda_D, \cdots, (\lambda_D + 1) \cdot 3^{n^i}\}$ such that

$$s_i(t; s_i(0) = s_{i_0}) = s_\omega, \ \forall\, t \geqslant T_\omega^i, \ t \in \mathbb{N}, \quad \text{(A.11)}$$

that is, network $\Xi^i$ achieves finite-time weak cooperation in time $T_\omega^i$ with $s_i(0) := s_{i_0}$. The proof is completed. ■

## Appendix B. PROOF OF THEOREM 6

Since $\Xi^i$, $i \in \{1, 2\}$, achieves finite strong cooperation at time $T_i \in \mathbb{N}$, based on Definition 6, there does not exist $t \in \mathbb{N}$ to satisfying $\Psi_{p_\star^i}(t) = 0$. According to (9), (10) and (11), the strategy profile of $\Xi^i$ can be expressed as

$$s_i(t+1) := \mathbb{G}^i s_i(t), \ \forall\, t \in \mathbb{N}.$$

Moreover,

$$(\mathbb{G}^i)^t s_{i_0} = s_e^i, \ \forall\, t \geqslant T_i, \ t \in \mathbb{N} \quad \text{(B.1)}$$

holds for any initial strategy profile $s_i(0) := s_{i_0}$. Applying the Cayley-Hamilton theorem to (B.1), there exists a positive integer $T_e^i < 3^{n^i}$ such that

$$(\mathbb{G}^i)^t s_{i_0} = s_e^i, \ \forall\, t \geqslant T_e^i, \ t \in \mathbb{N}$$

holds for any $s_i(0) := s_{i_0}$, that is, network $\Xi^i$ achieves finite-time strong cooperation in time $T_e^i$. The proof is completed. ■